Green function for a two-dimensional discrete Laplace-Beltrami operator

Volodymyr Sushch

Koszalin University of Technology
Sniadeckich 2, 75-453 Koszalin, Poland
volodymyr.sushch@tu.koszalin.pl

Abstract

We study a discrete model of the Laplacian in $\mathbb{R}^2$ that preserves the geometric structure of the original continual object. This means that, speaking of a discrete model, we do not mean just the direct replacement of differential operators by difference ones but also a discrete analog of the Riemannian structure. We consider this structure on the appropriate combinatorial analog of differential forms. Self-adjointness and boundness for a discrete Laplacian are proved. We define the Green function for this operator and also derive an explicit formula of the one.

1 Introduction

We begin with a brief review of some well known definitions that are related to the contents of this paper. Denote by $\Lambda^r(\mathbb{R}^2)$ the set of all differentiable complex-valued r-forms on $\mathbb{R}^2$, where $r = 0, 1, 2$. Let $* : \Lambda^r(\mathbb{R}^2) \to \Lambda^{2-r}(\mathbb{R}^2)$ be the Hodge star operator. An inner product for r-forms with compact support is defined by

$$(\varphi, \ \psi) = \int\limits_{\mathbb{R}^2} \varphi \wedge *\overline{\psi}, \tag{1.1}$$

where the bar over ψ denotes complex conjugation. Let $L^2\Lambda^r(\mathbb{R}^2)$ denote the completion of $\Lambda^r(\mathbb{R}^2)$ with respect to the norm generated by the inner product (1.1). Let the exterior derivative $d : \Lambda^r(\mathbb{R}^2) \to \Lambda^{r+1}(\mathbb{R}^2)$ be defined as usual. We define the operator

$$d : L^2\Lambda^r(\mathbb{R}^2) \to L^2\Lambda^{r+1}(\mathbb{R}^2) \tag{1.2}$$

1

as the closure in the L^2-norm the corresponding operation specified on smooth forms. The adjoint of d, denoted by δ, is given by

$$(d\varphi,\ \omega) = (\varphi,\ \delta\omega), \qquad \varphi \in L^2\Lambda^r(\mathbb{R}^2), \quad \omega \in L^2\Lambda^{r+1}(\mathbb{R}^2).$$

Note that $\delta : L^2\Lambda^{r+1}(\mathbb{R}^2) \to L^2\Lambda^r(\mathbb{R}^2)$. The following relations hold among $*$, d and δ. See for instance [15].

$$*^2 = (-1)^{r(2-r)}Id, \qquad \delta = (-1)^r *^{-1} d *. \tag{1.3}$$

The Laplacian is defined to be

$$-\Delta = d\delta + \delta d : L^2\Lambda^r(\mathbb{R}^2) \to L^2\Lambda^r(\mathbb{R}^2). \tag{1.4}$$

In this paper we develop some combinatorial structures that are analogs of objects in differential geometry. We are interested in finding of a natural discrete analog of the Laplacian on cochains. Speaking of a discrete model, we mean not only the direct replacement of differential operators by difference ones but also discrete analogs of all essential ingredients of the Riemannian structure over a properly introduced combinatorial object.

Our approach bases on the formalism proposed by Dezin [5]. We adapt the combinatorial constructions from [5, 11, 12] and define discrete analogs of operators (1.2)–(1.4) in a similar way. In [5], Dezin study discrete Laplace operators in finite-dimensional Hilbert spaces, i.e. on cochains given in domains with boundary. In this paper we extend these results on an infinite complex of complex-valued cochains. We prove self-adjointness and boundness for the discrete Laplace-Beltrami operator in infinite Hilbert spaces that are associated with $L^2\Lambda^r(\mathbb{R}^2)$. Spectral properties are discussed. We define the Green function for the discrete Laplacian and derive the one in an explicit form.

There are other geometric approaches to discretisation of the Hodge theory of harmonic forms presented in [6, 7, 8]. In all these papers discrete models are given on the simplicial cochains of triangulated closed Riemannian manifolds. See also [3, 4, 9] and references given there.

Classical references on second order difference equations are the books by Berezanski [2, ch. 7], Atkinson [1] and the most recent monograph by Teschl [14].

2   Combinatorial structures

Let us denote by $\mathfrak{C}(2)$ the two-dimensional complex. This complex is defined by

$$\mathfrak{C}(2) = \mathfrak{C}^0 \oplus \mathfrak{C}^1 \oplus \mathfrak{C}^2,$$

where $\mathfrak{C}^r$ is a real linear space of r-dimensional chains. We follow the notation of [5, 12]. Let $\{x_{k,s}\}$, $\{e^1_{k,s},\ e^2_{k,s}\}$, $\{\Omega_{k,s}\}$, $k, s \in \mathbb{Z}$, be the sets of basis elements of $\mathfrak{C}^0$, $\mathfrak{C}^1$, $\mathfrak{C}^2$ respectively. It is convenient to introduce shift operators

$$\tau k = k + 1, \qquad \sigma k = k - 1$$

in the set of indices. The boundary operator ∂ is defined by the rule

$$\partial x_{k,s} = 0, \quad \partial\Omega_{k,s} = e^1_{k,s} + e^2_{\tau k,s} - e^1_{k,\tau s} - e^2_{k,s},$$
$$\partial e^1_{k,s} = x_{\tau k,s} - x_{k,s}, \quad \partial e^2_{k,s} = x_{k,\tau s} - x_{k,s}. \tag{2.1}$$

The definition of ∂ is linearly extended to arbitrary chains. We call the complex $\mathfrak{C}(2)$ a combinatorial model of $\mathbb{R}^2$.

On the other hand, we can consider $\mathfrak{C}(2)$ as the tensor product $\mathfrak{C}(2) = \mathfrak{C} \otimes \mathfrak{C}$ of the one-dimensional complex $\mathfrak{C}$ (combinatorial model of a real line). Then basis elements of $\mathfrak{C}(2)$ can be written as follows

$$x_k \otimes x_s = x_{k,s}, \qquad e_k \otimes x_s = e^1_{k,s},$$
$$e_k \otimes e_s = \Omega_{k,s}, \qquad x_k \otimes e_s = e^2_{k,s},$$

where x_k, e_k are the basis elements of $\mathfrak{C}$.

Let us introduce an object dual to $\mathfrak{C}(2)$. Namely, the complex of complex-valued functions over $\mathfrak{C}(2)$. The dual complex $K(2)$ we can consider as the set of complex-valued cochains and it has the same structure as $\mathfrak{C}(2)$, i.e. $K(2) = K^0 \oplus K^1 \oplus K^2$. In other words, $K(2)$ is a linear complex space with basis elements

$$\{x^{k,s}, \ e^{k,s}_1, \ e^{k,s}_2, \ \Omega^{k,s}\}.$$

The pairing (chain-cochain) operation is defined by the rule:

$$< x_{k,s}, \ x^{p,q} > = < \Omega_{k,s}, \ \Omega^{p,q} > = < e^1_{k,s}, \ e^{p,q}_1 > = < e^2_{k,s}, \ e^{p,q}_2 > = \delta^{p,q}_{k,s}, \quad (2.2)$$

where $\delta^{p,q}_{k,s}$ is Kronecker symbol. We call elements of the complex $K(2)$ forms, emphasizing their closeness to the corresponding continual objects, differential forms. Then the 0-, 1-, 2-forms φ, $\omega = (u,v)$, η can be written as

$$\varphi = \sum_{k,s} \varphi_{k,s} x^{k,s}, \qquad \eta = \sum_{k,s} \eta_{k,s}\Omega^{k,s}, \qquad \omega = \sum_{k,s}(u_{k,s}e^{k,s}_1 + v_{k,s}e^{k,s}_2), \quad (2.3)$$

where $\varphi_{k,s}, \ u_{k,s}, \ v_{k,s}, \ \eta_{k,s} \in \mathbb{C}$ for any $k, s \in \mathbb{Z}$. Operation (2.2) is extended to arbitrary forms (2.3) by linearity. The boundary operator (2.1) in $\mathfrak{C}(2)$ induces the dual operation d^c in $K(2)$:

$$< \partial a, \ \alpha > = < a, \ d^c\alpha >, \tag{2.4}$$

where $a \in \mathfrak{C}(2)$, $\alpha \in K(2)$. We assume that the coboundary operator $d^c : K^r \to K^{r+1}$ is a discrete analog of the exterior differentiation operator d (1.3).

If $\varphi \in K^0$ and $\omega = (u,v) \in K^1$, then we have the following difference representations for d^c:

$$< e^1_{k,s}, \ d^c\varphi > = \varphi_{\tau k,s} - \varphi_{k,s} \equiv \Delta_k\varphi_{k,s},$$
$$< e^2_{k,s}, \ d^c\varphi > = \varphi_{k,\tau s} - \varphi_{k,s} \equiv \Delta_s\varphi_{k,s}$$
$$< \Omega_{k,s}, \ d^c\omega > = v_{\tau k,s} - v_{k,s} - u_{k,\tau s} + u_{k,s} \equiv \Delta_k v_{k,s} - \Delta_s u_{k,s}. \tag{2.5}$$

Note that if $\eta \in K^2$, then $d^c\eta = 0$.

Let us now introduce in $K(2)$ a multiplication which is an analog of the exterior multiplication $\wedge$ for differential forms. We denote this operation by $\cup$ and define it according to the rule:

$$x^{k,s} \cup x^{k,s} = x^{k,s}, \qquad e_2^{k,s} \cup e_1^{k,\tau s} = -\Omega^{k,s},$$
$$x^{k,s} \cup e_1^{k,s} = e_1^{k,s} \cup x^{\tau k,s} = e_1^{k,s},$$
$$x^{k,s} \cup e_2^{k,s} = e_2^{k,s} \cup x^{k,\tau s} = e_2^{k,s},$$
$$x^{k,s} \cup \Omega^{k,s} = \Omega^{k,s} \cup x^{\tau k, \tau s} = e_1^{k,s} \cup e_2^{\tau k,s} = \Omega^{k,s}, \qquad (2.6)$$

supposing the product to be zero in all other cases. The $\cup$-multiplication is extended to discrete forms by linearity. In terms of the theory of homologies, this is the so-called Whitney multiplication. For arbitrary forms $\alpha, \beta \in K(2)$ we have the following relation

$$d^c(\alpha \cup \beta) = d^c\alpha \cup \beta + (-1)^r\alpha \cup d^c\beta, \qquad (2.7)$$

where r is the degree of α. The proof of this can be found in Dezin [5, p. 147]. Relation (2.7) is an analog of the corresponding continual relation for differential forms (see [15]).

Define a discrete analog of the Hodge star operator. Let $\varepsilon^{k,s}$ denote an arbitrary basis element of $K(2)$. We introduce the operation $* : K^r \to K^{2-r}$ by setting

$$\varepsilon^{k,s} \cup *\varepsilon^{k,s} = \Omega^{k,s}. \qquad (2.8)$$

Using (2.6) we get

$$*x^{k,s} = \Omega^{k,s}, \quad *e_1^{k,s} = e_2^{\tau k,s}, \quad *e_2^{k,s} = -e_1^{k,\tau s}, \quad *\Omega^{k,s} = x^{\tau k, \tau s}.$$

The operation $*$ is extended to arbitrary forms by linearity.

Let $\alpha \in K^r$ is an arbitrary r-form:

$$\alpha = \sum_{k,s} \alpha_{k,s}\varepsilon^{k,s}. \qquad (2.9)$$

Denote by K_0^r the set of all discrete r-form with compact support on $\mathfrak{C}(2)$. Let Ω be the following "domain"

$$\Omega = \sum_{k,s} \Omega_{k,s}, \qquad k, s \in \mathbb{Z}, \qquad (2.10)$$

where $\Omega_{k,s}$ is a two-dimensional basis element of $\mathfrak{C}(2)$. Note that if the sum (2.10) is finite and let $-N \leq k, s \leq N$, $N \in \mathbb{N}$, then we will write $\Omega = \Omega_N$.

The relation

$$(\alpha, \ \beta) = < \Omega, \ \alpha \cup *\overline{\beta} >, \qquad (2.11)$$

4

where $\alpha, \beta \in K_0^r$, gives a correct definition of inner product in $K(2)$. Using (2.2), (2.6) and (2.8), this definition can be rewritten as follows

$$(\alpha, \ \beta) = \sum_{k,s} \alpha_{k,s} \overline{\beta_{k,s}}. \tag{2.12}$$

For $\Omega = \Omega_N$ we will write

$$(\alpha, \ \beta)_N = < \Omega_N, \ \alpha \cup * \overline{\beta} >= \sum_{k,s=-N}^{N} \alpha_{k,s} \overline{\beta_{k,s}}.$$

Let $\alpha \in K^r$, $\beta \in K^{r+1}$. The relation

$$(d^c \alpha, \ \beta)_N = < \partial \Omega_N, \ \alpha \cup * \overline{\beta} > +(\alpha, \ \delta^c \beta)_N \tag{2.13}$$

defines the operator $\delta^c : K^{r+1} \rightarrow K^r$,

$$\delta^c \beta = (-1)^r *^{-1} d^c * \beta,$$

which is the formally adjoint operator to d^c (see [5] for more details). It is obvious that the operator δ^c can be regarded as a discrete analog of the codifferential δ (cf. (1.3)). Equation (2.13) is an analog of the Green formula for the formally adjoint differential operators d and δ. It is easy to check that for $\alpha \in K_0^r$, $\beta \in K_0^{r+1}$ we obtain

$$(d^c \alpha, \ \beta) = (\alpha, \ \delta^c \beta). \tag{2.14}$$

According to (2.5), we have $\delta^c \varphi = 0$ and

$$< x_{k,s}, \ \delta^c \omega >= -\Delta_k u_{\sigma k,s} - \Delta_s v_{k,\sigma s}, \tag{2.15}$$

$$< e_{k,s}^1, \ \delta^c \eta >= \Delta_s \eta_{k,\sigma s}, \qquad < e_{k,s}^2, \ \delta^c \eta >= -\Delta_k \eta_{\sigma k,s}, \tag{2.16}$$

where $\varphi \in K^0$, $\omega \in K^1$ and $\eta \in K^2$.

Therefore a discrete analog of the Laplace-Beltrami operator (1.4) can be defined as follows

$$- \Delta^c = \delta^c d^c + d^c \delta^c : K^r \rightarrow K^r. \tag{2.17}$$

Obviously, if $\varphi \in K^0$, then we have

$$- \Delta^c \varphi = \delta^c d^c \varphi. \tag{2.18}$$

Combining (2.15) with (2.5) we can rewrite (2.18) as

$$< x_{k,s}, \ -\Delta^c \varphi >= 4\varphi_{k,s} - \varphi_{\tau k,s} - \varphi_{k,\tau s} - \varphi_{\sigma k,s} - \varphi_{k,\sigma s}. \tag{2.19}$$

The same difference form of (2.17) can be drawn for the components $\eta_{k,s}$ of $\eta \in K^2$ and for the two components $u_{k,s} \ v_{k,s}$ of $\omega \in K^1$.

3 Discrete Laplacian

Let us now introduce the linear space

$$\mathcal{H}^r = \{\alpha \in K^r : \sum_{k,s} |\alpha_{k,s}|^2 < +\infty, \quad k, s \in \mathbb{Z}\}, \tag{3.1}$$

where $r = 0, 1, 2$. Clearly, $\mathcal{H}^r$ is a Hilbert space with inner product (2.11) (or (2.12)) and with the following norm

$$\|\alpha\| = \sqrt{(\alpha, \ \alpha)} = \Big(\sum_{k,s} |\alpha_{k,s}|^2 \Big)^{\frac{1}{2}}. \tag{3.2}$$

Note that if $\alpha \in \mathcal{H}^r$, then the set of complex-valued sequences $(\alpha_{k,s})$ is $\ell^2(\mathbb{Z}^2)$. From now on we regard d^c, δ^c and $-\Delta^c$ as the following operators

$$d^c : \mathcal{H}^r \to \mathcal{H}^{r+1}, \qquad \delta^c : \mathcal{H}^r \to \mathcal{H}^{r-1}, \qquad -\Delta^c : \mathcal{H}^r \to \mathcal{H}^r,$$

where $r = 0, 1, 2$. It is convenient to suppose that $\mathcal{H}^{-1} = \mathcal{H}^3 = 0$.

Theorem 1. *The operators*

$$-\Delta^c : \mathcal{H}^r \to \mathcal{H}^r, \qquad r = 0, 1, 2, \tag{3.3}$$

are bounded and self-adjoint. Moreover, $\| - \Delta^c \| = 8$, where $\| - \Delta^c \|$ denotes the operator norm of $-\Delta^c$.

Proof. We begin by proving self-adjointness of $-\Delta^c$ for the case $r = 0$. Let $\varphi, \psi \in \mathcal{H}^0$ and $\omega = (u, v) \in \mathcal{H}^1$. Then the Green formula (2.13) can be rewritten as

$$(d^c \varphi, \ \omega)_N = \sum_{k=-N}^{N} (\varphi_{k,\tau N} \overline{v_{k,N}} - \varphi_{k,-N} \overline{v_{k,-\tau N}}) +$$

$$+ \sum_{s=-N}^{N} (\varphi_{\tau N,s} \overline{u_{N,s}} - \varphi_{-N,s} \overline{u_{-\tau N,s}}) + (\varphi, \ \delta^c \omega)_N \tag{3.4}$$

The substitution of $d^c \psi$ for ω in (3.4) gives

$$(d^c \varphi, \ d^c \psi)_N = \sum_{k=-N}^{N} \Big(\varphi_{k,\tau N} (\overline{\psi_{k,\tau N}} - \overline{\psi_{k,N}}) - \varphi_{k,-N} (\overline{\psi_{k,-N}} - \overline{\psi_{k,-\tau N}}) \Big) +$$

$$+ \sum_{s=-N}^{N} \Big(\varphi_{\tau N,s} (\overline{\psi_{\tau N,s}} - \overline{\psi_{N,s}}) - \varphi_{-N,s} (\overline{\psi_{-N,s}} - \overline{\psi_{-\tau N,s}}) \Big) + (\varphi, \ \delta^c d^c \psi)_N. \tag{3.5}$$

Letting $N \to +\infty$ we get

$$(d^c \varphi, \ d^c \psi) = (\varphi, \ \delta^c d^c \psi). \tag{3.6}$$

It follows immediately that $-\Delta^c : \mathcal{H}^0 \to \mathcal{H}^0$ is self-adjoint.

The same proof remains valid for the case $r = 2$. Now we have $-\Delta^c = d^c \delta^c$ and the analog of relation (3.5) results from the inner product $(\delta^c \eta, \ \delta^c \zeta)_N$, where $\eta, \zeta \in \mathcal{H}^2$. A trivial verification shows that properties of $(\delta^c \eta, \ \delta^c \zeta)_N$ are completely similar to those of $(d^c \varphi, \ d^c \psi)_N$. Hence

$$(\delta^c \eta, \ \delta^c \zeta) = (\eta, \ d^c \delta^c \zeta). \tag{3.7}$$

Finally, let $r = 1$. In this case we have $-\Delta^c = d^c \delta^c + \delta^c d^c$ and we must study the sum $(d^c \omega, \ d^c \vartheta) + (\delta^c \omega, \ \delta^c \vartheta)$, where $\omega = (u, v) \in \mathcal{H}^1$, $\vartheta = (f, g) \in \mathcal{H}^1$. Taking in (2.13) $\alpha = \omega$ and $\beta = d^c \vartheta$ we obtain the analog of relation (3.5) for 1-forms

$$(d^c \omega, \ d^c \vartheta)_N \equiv \sum_{k,s=-N}^{N} (\Delta_k v_{k,s} - \Delta_s u_{k,s})(\overline{\Delta_k g_{k,s} - \Delta_s f_{k,s}}) =$$

$$= \sum_{k=-N}^{N} \left[u_{k,-N}(\overline{\Delta_k g_{k,-\tau N}} - \overline{\Delta_N f_{k,-\tau N}}) - u_{k,\tau N}(\overline{\Delta_k g_{k,N}} - \overline{\Delta_N f_{k,N}}) \right] +$$

$$+ \sum_{s=-N}^{N} \left[v_{\tau N,s}(\overline{\Delta_N g_{N,s}} - \overline{\Delta_s f_{N,s}}) - v_{-N,s}(\overline{\Delta_N g_{-\tau N,s}} - \overline{\Delta_s f_{-\tau N,s}}) \right] +$$

$$+ (\omega, \ \delta^c d^c \vartheta)_N.$$

Letting $N \to +\infty$ we obtain equation (3.6) for the 1-forms $\omega, \vartheta \in \mathcal{H}^1$. In the same manner we can see that equation (3.7) holds for $\omega, \vartheta \in \mathcal{H}^1$. Adding we obtain

$$(d^c \omega, \ d^c \vartheta) + (\delta^c \omega, \ \delta^c \vartheta) = (\omega, \ -\Delta^c \vartheta). \tag{3.8}$$

Thus it follows that

$$(-\Delta^c \omega, \ \vartheta) = (\omega, \ -\Delta^c \vartheta).$$

For the rest of the proof let $\alpha \in \mathcal{H}^r$ be an arbitrary r-form. Substituting (2.19) into (2.12) we get

$$|(-\Delta^c \alpha, \ \alpha)| = \left| \sum_{k,s} (4\alpha_{k,s} - \alpha_{\tau k,s} - \alpha_{k,\tau s} - \alpha_{\sigma k,s} - \alpha_{k,\sigma s}) \overline{\alpha_{k,s}} \right| \leq$$

$$\leq 4 \sum_{k,s} |\alpha_{k,s}|^2 + \sum_{k,s} |\alpha_{\tau k,s} \overline{\alpha_{k,s}}| + \sum_{k,s} |\alpha_{k,\tau s} \overline{\alpha_{k,s}}| +$$

$$+ \sum_{k,s} |\alpha_{\sigma k,s} \overline{\alpha_{k,s}}| + \sum_{k,s} |\alpha_{k,\sigma s} \overline{\alpha_{k,s}}| \leq 8 \|\alpha\|^2.$$

From this we conclude that $\| -\Delta^c \| \leq 8$. Since $-\Delta^c$ is self-adjoint, it follows easily that $\| -\Delta^c \| = 8$ (see for instance [10, Ch. 3]). $\qquad \square$

Corollary 2. *The operators (3.3) are positive, i.e. for any non-trivial r-form $\alpha \in \mathcal{H}^r$ we have*

$$(-\Delta^c \alpha, \ \alpha) > 0.$$

Proof. This follows from (3.6), (3.7) and (3.8). □

Corollary 3. *For any r, $r = 0, 1, 2$, we have*

$$\sigma(-\Delta^c) = [0, \ 8],$$

where $\sigma(-\Delta^c)$ denotes the spectrum of $-\Delta^c$.

Proof. Straightforward. □

4 Discrete analog of the Green function

Let $\varrho(-\Delta^c) = \mathbb{C} \setminus \sigma(-\Delta^c)$ denotes the resolvent set of $-\Delta^c$. In this section we try to describe the resolvent operator $(-\Delta^c - \lambda)^{-1}$, $\lambda \in \varrho(-\Delta^c)$, of the operator $-\Delta^c : \mathcal{H}^0 \to \mathcal{H}^0$. Let us introduce a discrete form $G(x, \tilde{x})$ on $\mathfrak{C}^0 \times \mathfrak{C}^0$ as follows

$$G(x, \tilde{x}) = \sum_{k,s} G_{k,s}(\tilde{x}) x^{k,s}, \quad \text{where} \quad G_{k,s}(\tilde{x}) = \sum_{m,n} G_{k,s,m,n} x^{m,n}$$

and $G_{k,s,m,n} \in \mathbb{C}$ for any $k, s, m, n \in \mathbb{Z}$. Hence we have

$$G(x, \tilde{x}) = \sum_{k,s} \sum_{m,n} G_{k,s,m,n} x^{k,s} x^{m,n}. \tag{4.1}$$

This is a so-called discrete double form (for details see [11]). Note that in the continual case (for differential forms) this construction is due to De Rham [13]. It is obvious that the basis elements $x^{k,s}$ and $x^{m,n}$ in (4.1) commute and $G(x, \tilde{x}) = G(\tilde{x}, x)$. By analogy with (2.2), the double 0-form $G(x, \tilde{x})$ can be written pointwise as

$$< x_{k,s} x_{m,n}, \ G(x, \tilde{x}) >= G_{k,s,m,n}. \tag{4.2}$$

Let $\varphi \in \mathcal{H}^0$. Then we define a 0-form $\delta^{m,n}$, $m, n \in \mathbb{Z}$, by setting

$$(\varphi, \ \delta^{m,n}) = \sum_{k,s} \varphi_{k,s} \delta_{k,s}^{m,n} = \varphi_{m,n}, \tag{4.3}$$

where $\delta_{k,s}^{m,n}$ is the Kronecker delta. By analogy with the continual case, the 0-form $\delta^{m,n}$ defined by (4.3) will be called a discrete analog of Dirac's δ-function at the point $x_{m,n}$. We can write $\delta^{m,n}$ as

$$\delta^{m,n} = \sum_{k,s} \delta_{k,s}^{m,n} x^{k,s} = x^{m,n}.$$

We need also the following double form

$$\delta(x, \tilde{x}) = \sum_{k,s} \sum_{m,n} \delta_{k,s}^{m,n} x^{k,s} x^{m,n} = \sum_{m,n} \delta^{m,n} x^{m,n} = \sum_{m,n} x^{m,n} x^{m,n}. \tag{4.4}$$

Definition 4. *The double form (4.1) is called the Green function for the operator* $-\Delta^c : \mathcal{H}^0 \to \mathcal{H}^0$ *if*

$$G_{k,s,m,n}(\lambda) = (\delta^{k,s}, \; (-\Delta^c - \lambda)^{-1}\delta^{m,n}) \tag{4.5}$$

for any $k, s, m, n \in \mathbb{Z}$.

Of course,

$$(-\Delta^c - \lambda)^{-1}\delta^{m,n} = G_{m,n}(x, \lambda).$$

It follows easily that

$$(-\Delta^c - \lambda)_x G(x, \tilde{x}, \lambda) = \delta(x, \tilde{x}), \tag{4.6}$$

where $(-\Delta^c - \lambda)_x$ is the operator $-\Delta^c - \lambda$ that acts with respect to x. Recall that in our abbreviation x corresponds to k, s.

We have

$$(-\Delta^c - \lambda)^{-1}\varphi = \sum_{k,s} \sum_{m,n} G_{k,s,m,n}(\lambda)\varphi_{m,n} x^{k,s} x^{m,n}, \qquad \varphi \in \mathcal{H}^0, \quad \lambda \in \varrho(-\Delta^c).$$

Indeed, applying $(-\Delta^c - \lambda)_x$ to the right-hand side gives

$$(-\Delta^c - \lambda)_x \sum_{m,n} \varphi_{m,n} G_{m,n}(x, \lambda) x^{m,n} = \sum_{m,n} \varphi_{m,n}(-\Delta^c - \lambda)_x G_{m,n}(x, \lambda) x^{m,n} =$$

$$= \sum_{m,n} \varphi_{m,n} \delta^{m,n} x^{m,n} = \sum_{m,n} \varphi_{m,n} x^{m,n} x^{m,n} = \varphi.$$

We now try to write the Green function for $-\Delta^c$ in a somewhat more explicit way. For this we construct a solution of the equation

$$-\Delta^c\varphi = \lambda\varphi, \qquad \lambda \in \mathbb{C}. \tag{4.7}$$

The following construction is adapted from [14], where the Green function is studied for Jacobi operators. By (2.19), equation (4.7) can be written pointwise (at the point $x_{k,s}$) as

$$4\varphi_{k,s} - \varphi_{\tau k,s} - \varphi_{k,\tau s} - \varphi_{\sigma k,s} - \varphi_{k,\sigma s} = \lambda\varphi_{k,s}. \tag{4.8}$$

Applying the transformation $\lambda = -4\mu + 4$ we reduce (4.8) to the equation

$$\frac{1}{4}(\varphi_{\tau k,s} + \varphi_{k,\tau s} + \varphi_{\sigma k,s} + \varphi_{k,\sigma s}) = \mu\varphi_{k,s}. \tag{4.9}$$

An easy computation shows that, substituting the ansatz $\varphi_{k,s} = p^{k+s}$ into (4.9), we obtain

$$\varphi^{\pm}_{k,s}(\mu) = (\mu \pm R(\mu))^{k+s}, \tag{4.10}$$

where $R(\mu) = -\sqrt{\mu^2 - 1}$ and $\sqrt{\cdot}$ denotes the standard brunch of the square root. It follows that we can write the solutions $\varphi^{\pm}(\lambda)$ of (4.7) in the form

$$\varphi^{\pm}(\lambda) = \sum_{k,s} \varphi^{\pm}_{k,s}(\mu) x^{k,s}, \tag{4.11}$$

where $\varphi_{k,s}^{\pm}(\mu)$ are given by (4.10) and $\mu = 1 - \frac{\lambda}{4}$. Obviously, $\lambda \in [0,\ 8]$ if and only if $\mu \in [-1,1]$. By Corollary 3, $\lambda \in \varrho(-\Delta^c)$ leads to $\mu \in \mathbb{C} \setminus [-1,1]$.

It is convenient to write $\varphi_{k,s}^{\pm}(\mu)$ as

$$\varphi_{k,s}^{\pm}(\mu) = \varphi_k^{\pm}(\mu) \cdot \varphi_s^{\pm}(\mu), \tag{4.12}$$

where $\varphi_k^{\pm}(\mu) = (\mu \pm R(\mu))^k$ and $k, s \in \mathbb{Z}$. Suppose $\mu \in \mathbb{C} \setminus [-1,1]$. Then one has to examine that the sequences $(\varphi_k^{\pm}(\mu))_{k \in \mathbb{Z}}$ are square summable near $\pm \infty$ respectively, i.e these are $\ell^2(\mathbb{Z})$ near $\pm \infty$.

Let $\mathcal{H}_{\pm}^0$ denotes the set of 0-forms whose restriction to $K_{\pm}^0$ belongs to $\mathcal{H}^0$. Here K_+^0 (K_-^0) denotes the set of 0-cochains (2.9) with $k + s > 0$ $(k + s < 0)$.

Proposition 5. *Let $\lambda \in \varrho(-\Delta^c)$. Then $\varphi^{\pm}(\lambda) \in \mathcal{H}_{\pm}^0$.*

Proof. This follows immediately from (4.12). $\qquad\square$

Lemma 6. *For any $k, s \in \mathbb{Z}$ the component $\varphi_k^+(\mu) \cdot \varphi_s^-(\mu)$ is a solution of (4.9)*

Proof. It is easy to check that

$$\varphi_2^{\pm} - 2\mu\varphi_1^{\pm} + 1 = 0.$$

It follows that $\varphi_k^+(\mu) \cdot \varphi_s^-(\mu)$ satisfies equation (4.9). Indeed, putting this in (4.9), we have

$$\varphi_{\sigma k}^+(\mu)\varphi_{\sigma s}^-(\mu)\big[(\varphi_2^+ - 2\mu\varphi_1^+ + 1)\varphi_1^- + (\varphi_2^- - 2\mu\varphi_1^- + 1)\varphi_1^+\big] = 0.$$

$\qquad\square$

Theorem 7. *Let $\lambda \in \varrho(-\Delta^c)$. Then the components (4.5) of the Green function are given by*

$$G_{k,s,m,n}(\lambda) = \frac{-1}{4R(\mu)} \begin{cases} \big(\mu + R(\mu)\big)^{|\tau k - m| + |\tau s - n|} & \text{for } k = m,\ s > n \\ & \text{or } k > m,\ s = n, \\ \big(\mu + R(\mu)\big)^{|\sigma k - m| + |\sigma s - n|} & \text{for } k = m,\ s < n \\ & \text{or } k < m,\ s = n, \\ \big(\mu + R(\mu)\big)^{|k - m| + |s - n|} & \text{for the all others,} \end{cases} \tag{4.13}$$

where $\mu = 1 - \frac{\lambda}{4}$.

Proof. We must prove that

$$< x_{k,s} x_{m,n},\ (-\Delta^c - \lambda)_x G(x, \tilde{x}) > = \delta_{k,s}^{m,n}, \tag{4.14}$$

where $G(x, \tilde{x})$ is given by (4.1). Using (2.19) and (4.1), we can rewrite the left-hand side of (4.14) as

$$(4 - \lambda)G_{k,s,m,n} - G_{\tau k,s,m,n} - G_{\sigma k,s,m,n} - G_{k,\tau s,m,n} - G_{k,\sigma s,m,n} =$$
$$= 4\mu G_{k,s,m,n} - (G_{\tau k,s,m,n} + G_{\sigma k,s,m,n} + G_{k,\tau s,m,n} + G_{k,\sigma s,m,n}). \tag{4.15}$$

The proof falls naturally into three parts.

Fix $x_{m,n}$. First, let $k = m$ and $s = n$. Substituting (4.13) into (4.15) and using (4.12), we obtain

$$< x_{k,s}x_{m,n}, \ (-\Delta^c - \lambda)_x G(x, \tilde{x}) > = \frac{1}{R(\mu)}(-\mu\varphi_0^+(\mu)\varphi_0^+(\mu) + \varphi_1^+(\mu)\varphi_0^+(\mu)) =$$

$$= \frac{-\mu + \varphi_1^+}{R(\mu)} = 1.$$

Note that $\varphi_0^{\pm}(\mu) = (\mu \pm R(\mu))^0 = 1$.

Now we show that (4.13) satisfies equation (4.8) for the cases $k = m, \ s \neq n$ and $k \neq m, \ s = n$. Check the case $k = m, \ s > n$. In this case the left-hand side of (4.14) is equal to

$$\frac{-1}{4R(\mu)}\left[4\mu\varphi_1^+(\mu)\varphi_{s-n+1}^+(\mu) - \varphi_2^+(\mu)\varphi_{s-n+1}^+(\mu) - \varphi_0^+(\mu)\varphi_{s-n+1}^+(\mu) - \right.$$

$$\left. - \varphi_1^+(\mu)\varphi_{s-n+2}^+(\mu) - \varphi_1^+(\mu)\varphi_{s-n}^+(\mu)\right] = \frac{\varphi_{s-n+1}^+(\mu)}{2R(\mu)}(\varphi_2^+ - 2\mu\varphi_1^+ + 1) = 0.$$

The proof for the case $k = m, \ s < n$ (or $k \neq m, \ s = n$) is similar.

Finely, let $k \neq m, s \neq n$. We give the proof only for the case $k > m, \ s < n$. For the other cases the proof runs similarly. In this case we have

$$G_{k,s,m,n}(\lambda) = \frac{1}{-4R(\mu)}\varphi_{k-m}^+(\mu)\varphi_{n-s}^+(\mu) = \frac{1}{-4R(\mu)}\varphi_k^+(\mu)\varphi_m^-(\mu)\varphi_n^+(\mu)\varphi_s^-(\mu).$$

Here we use that $\varphi_{-k}^+ = \varphi_k^-$ for any $k \in \mathbb{Z}$. By Lemma 6, $\varphi_k^+(\mu)\varphi_s^-(\mu)$ is a solution of (4.9) and so is $G_{k,s,m,n}(\lambda)$ at the point $x_{k,s}$.

Thus, since $G(x, \tilde{x}, \lambda)$ with components given by (4.13) and, by Proposition 5, $G(x, \tilde{x}, \lambda) \in \mathcal{H}^0$ with respect to x, it must be the Green function of $-\Delta^c : \mathcal{H}^0 \to \mathcal{H}^0$. □

It should be noted that the consideration of $-\Delta^c : \mathcal{H}^2 \to \mathcal{H}^2$ does not differ from that carried out for the 0-forms $\varphi \in \mathcal{H}^0$. In this case we consider (4.1) as a double form on $\mathfrak{C}^2 \times \mathfrak{C}^2$ with basis elements $\Omega^{k,s}\Omega^{p,q}$. Then we define the Green function as above and its components are given by (4.13). The situation with $-\Delta^c : \mathcal{H}^1 \to \mathcal{H}^1$ is more difficult. In this case, having written equation (4.7) pointwise (at the elements $e_{k,s}^1$ and $e_{k,s}^2$), we obtain a pair of equations (4.8) which corresponds to the components $u_{k,s}, \ v_{k,s}$ of $\omega \in \mathcal{H}^1$. Roughly speaking, here we must describe the Green function for each component of the 1-form $\omega = (u, v)$. The reader can verify that the components $G_{k,s,m,n}(\lambda)$ of the Green function have again the form (4.13).

References

[1] F. Atkinson, *Discrete and continuous boundary problems*, Academic Press, New York, 1964.

[2] Yu. M. Berezanski, *Expansions in eigenfunctions of self-adjoint operators*, Transl. Math. Monographs, vol. **17**, Amer. Math. Soc., Providence, R.I., 1968.

[3] A. I. Bobenko, B. A. Springborn, *A discrete Laplace-Beltrami operator for simplicial surfaces*, arxiv: math.DG/0503219.

[4] F. Chung, S.-T. Yau, *Discrete Green's Functions*, J. Combin. Theory, Ser. A, **91**, Issue: 1-2, (2000), 191-214.

[5] A. A. Dezin, *Multidimensional analysis and discrete models*, CRC Press, Boca Raton, 1995.

[6] J. Dodziuk, *Finite-difference approach to Hodge theory of harmonic forms*, Amer. J. Math., **98**, (1976), 79–104.

[7] J. Dodziuk, V. K. Patodi, *Riemannian structures and triangulations of manifolds*, Journal of Indian Math. Soc., **40**, (1976), 1–52.

[8] J. Komorowski, *On finite-dimensional approximations of the exterior differential, codifferential and Laplacian on a Riemannian manifold*, Bull. Acad. Pol. Sci., **23**, No **9**, (1975), 999–1005.

[9] T. Mantuano, *Discretization of compact Riemannian manifolds applied to the spectrum of Laplacian*, Ann. Global Anal. Geom., **27**, (2005), 33-46.

[10] S. Mizohata, *The theory of partial differential equations*, Cambridge University Press, 1973.

[11] V. Sushch, *A discrete analogue of the Cauchy integral*, Differential Equations, **29**, (1993), No. 8, 1242–1249.

[12] V. Sushch, *On some discrete model of magnetic Laplacian*, Ukr. Math. Bull., **2**, (2005), No. 4, 591–607.

[13] G. De Rham, *Variétés différentiables*, Hermann, Paris, 1955.

[14] G. Teschl, *Jacobi operators and completely integrable nonlinear lattices*, Math. Surveys and Monographs, vol. **72**, Amer. Math. Soc., Providence, R.I., 2000.

[15] C. Von Westenholz, *Differential forms in mathematical physics*, North-Holland Publishing Company, Amsterdam, 1981.